Asia Pacific Journal of Research in Business Management
Vol. 8, Issue 1, January 2017 Impact Factor: 5.16, ISSN: (2229-4104)
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THE FORMULATION AND SOLUTION OF OPTIMAL CONSUMPTION PROBLEM DESCRIBED BY CONTINUOUS TIME MODEL

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ABSTRACT:

The formulation and viscosity solutions of the optimal consumption problem and the definition of the value function were carried out in this work.

Keywords: optimal consumption, viscosity solutions, Hamilton-Jacobi-Bellman equation,

INTRODUCTION:

The main objective of any investor is to maximize the total expected discounted utility of consumption. An optimal consumption problem help the investors to achieve their goal since it describe the optimal way that the investor can invest his many (save the money in a risk free bonds, invest it on the risky stock market or spend the money on consumption), see [2]. The concept of the viscosity solutions appear firstly in the work of Michael G. Crandall and Pierre-Louis Lions in 1983 see [1], the viscosity solution can be applied to the study of linear and nonlinear Partial differential equations of any order. Throughout this work we will consider a continuous-time consumption investment model.

DEFINITION:

The Hamilton–Jacobi–Bellman (HJB) equation is a partial differential equation which is central to optimal control theory. The solution of the HJB equation is the 'value function' which gives the minimum cost for a given dynamical system with associated cost function.

$$F\left(t, x, D_x V(t,x), D_{xx} V(t,x)\right) = -V_t(t,x) - \sup_{u \in U}\left\{\varphi(t,x,u) + L^u V(t,x)\right\} = 0 \qquad (1)$$

On $[0, T) \times R^n$.

DEFINITION:

A stochastic process S_t is said to follow a geometric Brownian motion (GBM) if it satisfies the following stochastic differential equation:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \qquad (2)$$

Where α and σ represent the mean return rate of the stock and the volatility of the stock respectively.

FORMULATION OF AN OPTIMAL CONSUMPTION PROBLEM:

For the formulation of the optimal consumption problem firstly let us consider the filtered probability space $(\Omega, F, \{F_t\}_{t\in T}, P)$ where, Ω represent the set of all possible outcomes , F is the σ-algebra of events, $\{F_t\}_{t\in T}$ denotes the information generated by the process over the time interval $[0,T]$ and P is the probability measure on which we will define the stochastic process $W(t)$ to be the standard 1- dimensional wiener process or Brownian motion.

Consider also the financial market with two assets stock and bond, and we will assume that the price of the bond is driven by an ordinary differential equation

$$d\,p_t = r\,p_t\,d\,t \tag{3}$$

Where r the risk-free interest rate.

And assume that the dynamics of the price asset S_t fellow the geometric Brownian motion.

The interest rate r , the mean rate of return α, and the volatility σ are assumed to be constant with $r > 0$, $\sigma > 0$, then it is well-known that the wealth process X_t of the investor is driven by:

$$d\,X_t = (r\,X_t + \pi_t\,\mu - C_t)d\,t + \sigma\,\pi_t\,dW_t \tag{4}$$

Where $\mu = \alpha - r > 0$, π_t is the amount of money invested in the stock at time t and C_t is the cumulative consumption at time t.

Since the target of any investor is to maximize the total expected (discounted) Utility from consumption over an infinite trading horizon, so the investor must choose the best consumption investment strategy $(C(.)\ ,\pi(.)\ )$ if so the optimal consumption problem can be defined as:

$$\max\ E\left[\int_0^\infty e^{-\beta t}\,U\,(C_t)\,d\,t\right] \tag{5}$$

Where $U : R^+ \to R^+$ is utility function of the investor and $\beta > 0$ is a constant discounting factor.

Define the value function $V(x)$ as:

$$V(x) = \sup_{(C(,), \pi(.))} E\left[\int_0^\infty e^{-\beta t} V(C_t)\,dt\right] \quad (6)$$

If $k > 0$ and $0 < p < 1$ then the value function $V(.)$ of problem (6) satisfy satisfies

$$V(x) \le \frac{1}{p} k^{p-1} x^p \quad , \; x > 0$$

THEOREM:

For some constants $0 < p < 1$ and $k > 0$, then the value function $V(x)$ of problem (6) is the unique viscosity solution of its associated HJB equation:

$$\beta V(x) - \sup_\pi \left(\frac{1}{2}\sigma^2\pi^2 V_{xx}(x) + \pi\mu V_x(x)\right)$$

$$-\sup_{0<C\le kx+c}\left(u(c) - C V_x(x)\right) - rx V_x(x)$$

$$= \beta V(x) + \frac{\mu^2 V_x^2(x)}{2\sigma^2 V_{xx}(x)} + \left(C(x) - rx\right)V_x(x) - \frac{C^p(x)}{p} = 0 \;\; . x > 0 \quad (7)$$

In the class of increasing concave function on $[0, +\infty)$ with $V(0) = 0$

$$C(x) = \min\left\{ V_x(x)^{\frac{1}{p-1}}(x), kx + l \right\}, x > 0$$

THEOREM:

If $k > 0$, $K > 0$ and $C = 0$ then the optimal consumption investment strategy for problem (6) is given by:

$$(C_t, \pi_t) = \left(\min(k, K)\, X_t \,,\, \frac{\mu}{\sigma^2 (1-p)} X_t\right), \; t \ge 0$$

And the optimal value defined as:

$$V(x) = \frac{\min(k,K)^p}{p\left(k(1-p)\right) + \min(k,K)^p} x^p$$

$$= \begin{cases} \dfrac{K^p}{p\left(k(1-p)\right) + K\,p} x^p & , K < k \\[2ex] \dfrac{1}{p} R^{p-1} x^p & , K \geq k \end{cases}$$

Finally we get that the value function of an optimal consumption problem is the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equation.

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