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MATHEMATICAL SOLUTION OF DIFFUSION EQUATION BY A METHOD OF LINES DISCRETISATION'S AND STABILITY THEORY OF FINITE DIFFERENCES

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Abstract:

The development of numerical techniques capable of quickly and accurately approximating the solution of partial differential equations remains one of the most difficult tasks in the field of applied mathematics. The Diffusion Equation is one of the most significant equations because of its use in hydraulic and transportation systems as well as the numerous challenges it frequently brings when solved numerically. In the current study, a Generalised Finite Differences scheme is used to offer a Method of Lines used to the numerical solution of the aforementioned problem in irregular regions.

Keywords:-Diffusion Equation, Lines Discretisation's, Stability

1. Introduction:

We'll now look at finite difference approaches for time-dependent partial differential equations, in which changes in space are linked to changes in time.

$$u_t = \kappa u_{xx}. \qquad (1)$$

This is the most common type of parabolic problem, and many of the general principles shown here can be applied to the development of numerical methods for other types of parabolic equations. For the sake of simplicity, we'll assume $\kappa = 1$, but we'll make some observations about how the results scale to various values of $\kappa > 0$. (If 0 is true, then (1) is an ill-posed problem

known as a "backward heat equation.") We also need initial circumstances at some point in time t_0, which we usually take to be $t_0 = 0$,

$u(x, 0) = \eta(x)$ (2)

If we're working on a bounded domain, we'll additionally need boundary conditions, such as the Dirichlet conditions.

$$u(0,t) = g_0(t) \text{ for } t > 0$$

$$u(1,t) = g_1(t) \text{ for } t > 0 \qquad (3) \text{if } 0 \leq x \leq 1.$$

In reality, we use a discrete grid with (x_i, t_n) grid points to solve a set of finite difference equations. where

$$x_i = ih,\ t_n = nk.$$

Here h = Δx k = Δt is the time step, and is the mesh spacing on the x-axis. Let $U_i^n \approx u(x_i, t_n)$ depict the grid point numerical approximation (x_i, t_n). We set up our difference equations in a way that we may march forward in time, because the heat equation is an evolution equation that can be solved forward in time, showing the values U_i^{n+1}i for all i from the values U_i^nat a previous time level, or even a multistep formula that includes values from earlier time levels. One natural discretization of (3), for example, would be

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n). \qquad (4)$$

This employs a forward difference in time and a standard centred difference in space. Because we can compute each value, this is an explicit method for U_i^{n+1}specifically in relation to the preceding data:

$$U_i^{n+1} = U_i^n + \frac{k}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n). \qquad (5)$$

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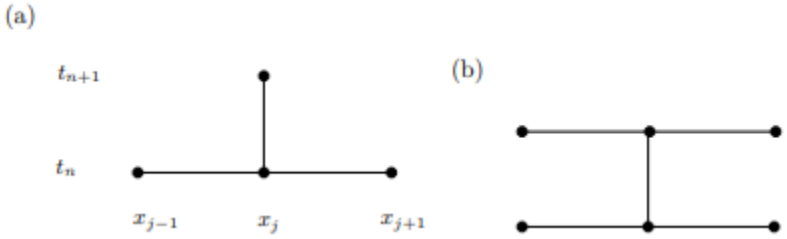

Figure 1: Stencils and for the methods

This method's stencil is shown in Figure 1. In the context of PDEs, this is a one-step technique in time, which is also known as a two-level approach because it involves the solution at two different time levels. The CrankNicolson approach is another one-step method that is far more beneficial in reality, as we will see later,

$$\begin{aligned}\frac{U_i^{n+1}-U_i^n}{k} &= \frac{1}{2}(D^2U_i^n + D^2U_i^{n+1}) \\ &= \frac{1}{2h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}),\end{aligned} \tag{6}$$

which is rewriteable as

$$U_i^{n+1} = U_i^n + \frac{k}{2h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) \tag{7}$$

$$-rU_{i-1}^{n+1} + (1+2r)U_i^{n+1} - rU_{i+1}^{n+1} = rU_{i-1}^n + (1-2r)U_i^n + rU_{i+1}^n \tag{8}$$

where r = k/2h^2 . This is an implicit method that results in a tridiagonal system of equations that must be solved for all values U_i^{n+1}simultaneously. This is how it looks in matrix form.

$$\begin{bmatrix} (1+2r) & -r & & & & & \\ -r & (1+2r) & -r & & & & \\ & -r & (1+2r) & -r & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -r & (1+2r) & -r & \\ & & & & -r & (1+2r) & \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ \vdots \\ U_{m-1}^{n+1} \\ U_m^{n+1} \end{bmatrix} \tag{9}$$

It's worth noting how the boundary conditions $u(0,t) = g_0(t)$ *and* $u(1,t) = g_1(t)$ are incorporated into these equations.This method is essentially as efficient per time step as an explicit method since a tridiagonal system of m equations can be solved with O(m) work. We'll see that the heat equation is "stiff," therefore this implicit method, which allows for far greater time steps than an explicit one, is a very efficient heat equation method. When using an implicit method to solve a parabolic equation, you must first solve a system of equations with the same structure as the 2-point boundary value issue. In each time step, a multidimensional parabolic equation also necessitates solving a problem with the structure of a multidimensional elliptic equation.When using an implicit method to solve a parabolic equation, you must first solve a system of equations with the same structure as the 2-point boundary value issue. In each time step, a multidimensional parabolic equation also necessitates solving a problem with the structure of a multidimensional elliptic equation.

2. Local truncation errors and order of accuracy

We can calculate the local truncation error by inserting the PDE's exact solution u(x,t) into the finite difference equation and determining how much it fails to satisfy the discrete equation.

Example: The method's local truncation error (6) is based on the form (7): $\mathcal{T}_i^n = \mathcal{T}(x_i, t_n), where$

$$\tau(x,t) = \frac{u(x,t+k) - u(x,t)}{k} - \frac{1}{h^2}(u(x-h,t) - 2u(x,t) + u(x+h,t)).$$

To achieve powers of k and h that agree with what we anticipate to observe in the global error, we should again utilise the form that directly represents the differential equation. Despite the fact that we don't know u(x,t) in general, we may find that if we assume it's smooth and utilise Taylor series expansions regarding u(x,t):

$$\tau(x,t) = \left(u_t + \frac{1}{2}ku_{tt} + \frac{1}{6}k^2u_{ttt} + \cdots\right) - \left(u_{xx} + \frac{1}{12}h^2u_{xxxx} + \cdots\right).$$

Since $u_t = u_{xx}$, the O(1) terms are dropped. By distinguishing $u_t = u_{xx}$ we find that $u_{tt} = u_{txx} = u_{xxxx}$ and so

$$\tau(x,t) = \left(\frac{1}{2}k - \frac{1}{12}h^2\right) u_{xxxx} + O(k^2 + h^4).$$

Because the truncation error is $O(h^2 + k)$, this approach is said to be second order accurate in space and first order accurate in time.In both space and time, the Crank-Nicolson approach is centred, and an investigation of its local truncation error reveals that it is second order accurate in both,

$$\tau\,(x,t) \;=\; O(k^2 + h^2)$$

Consistent method is said if τ (x,t) $\rightarrow$ 0 as k, h $\rightarrow$ 0. We assume that consistency, along with some form of stability, will be adequate to verify that the approach converges at each fixed point (X, T) as we refine the grid in both space and time, just as it has in the other examples we've looked at (boundary value problems and initial value problems for ODEs). Furthermore, we hope that the global order of accuracy for a stable approach will match with the order of the local truncation error, for example, for Crank-Nicolson we predict that:

$$U_i^n - u(X,T) = O(k^2 + h^2)$$

as k, h $\rightarrow$ 0 when ih $\equiv$ X and nk $\equiv$ T are fixed

After establishing the proper concept of stability, the fact that consistency plus stability is equivalent to convergence is known as the Lax Equivalence Theorem for linear PDEs. The hard (and intriguing) aspect of this theory, as usual, is defining and studying stability.

3. Method of Lines discretization's

To understand how stability theory for time-dependent PDE's connects to the stability theory we have already constructed for time-dependent ODE's, it is easiest to first analyse the so-called Method of Lines (MOL) discretization of the PDE. In this method, we discretize in space alone, resulting in a vast system of ODEs, each component of which corresponds to the solution at a

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grid point as a function of time. The system of ODEs can then be solved using one of the previously discussed methods for solving ODEs.

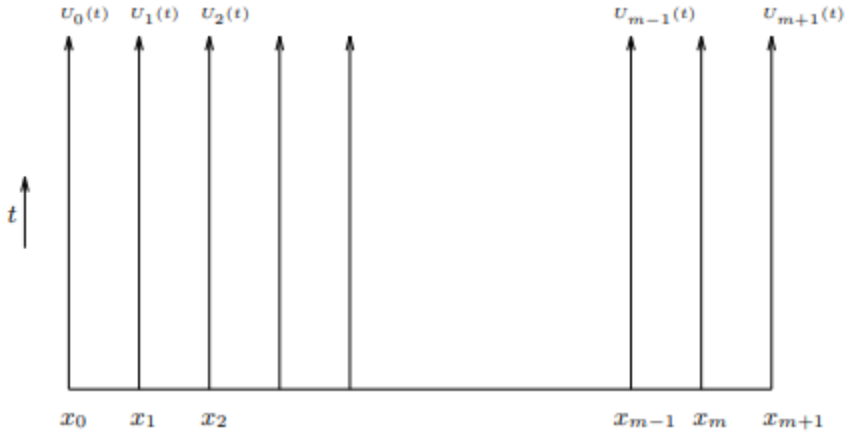

Figure 2: Interpretation method for lines. The solution along the line ahead in time at grid point x_i is $U_i(t)$.

For example, at grid point x_i, we could discretize the heat equation (9) in space by

$$U_i'(t) = \frac{1}{h^2}\left(U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)\right), \quad for i = 1,2,\ldots,m, \qquad (10)$$

where prime now denotes distinction in terms of time. This can be thought of as a coupled system of m ODEs for the variables $U_i(t)$, which vary continuously in time along the lines in Figure 2. This system can be expressed in the following way:

$$U'(t) = AU(t) + g(t) \qquad (11)$$

The terms needed for the boundary conditions are included in the tridiagonal matrix A and g(t), $U_0(t) \equiv g_0(t) and U_{m+1}(t) \equiv g_1(t)$

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$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \end{bmatrix}, \quad g(t) = \frac{1}{h^2}\begin{bmatrix} g_0(t) \\ 0 \\ 0 \\ \vdots \\ 0 \\ g_1(t) \end{bmatrix} \tag{12}$$

In reality, this MOL technique is sometimes applied by discretizing in space first and then using a software programme for systems of ODEs. There are additional packages built specifically for the application of MOL. This method has the advantage of being reasonably easy to apply to a wide range of time dependent PDEs, although it is generally less efficient than particularly devised approaches for the PDE.The MOL discretization, on the other hand, is tremendously useful as a tool for learning stability theory, and that is the main use we will make of it. The stability of ODE methods applied to a linear system of the type (12) depending on the eigenvalues of the matrix A, which now rely on the spatial discretization, is well understood. If we discretize the system using an ODE technique (12), We'll end up with a fully discrete approach that generates approximations $U_i^n \approx U_i(t_n)$ at discrete moments in time that are precisely the same as the points$(x_i, t_n)$of the grid we introduced earlier.

Applying Euler's approach, for example $U^{n+1} = U^n$ +kf(U^n) The fully discrete technique is applied to this linear system (12). A alternate technique might be obtained by using a higher order linear multistep or Runge-Kutta method, albeit the total method would only be second order accurate in space due to the spatial discretization (12). A more accurate technique could be obtained by replacing the right hand side of (12) with a higher-order approximation to $u_{xx}(x_i)$ and then utilising a higher order time discretization.

4. Stability theory

We can now look into the stability of techniques like (9) or (10), which may be thought of as conventional ODE methods applied to a linear system (12). We expect the approach to be stable if kλ∈ S, that is, if the time step k multiplied by any eigenvalue λ of A falls inside the ODE method's stability zone:

$$\lambda_p = \frac{2}{h^2}(cos(p\pi h) - 1, \quad forp = 1,2.,,,,.m, (13)$$

h = 1/(m+1), where m and h are connected once more. There is a new wrinkle here in comparison to the ODEs we considered the eigenvalues λ_ph Mesh Width h Mesh Width h Mesh Width h As we refine the grid and h → 0, the dimension of A grows, the number of eigenvalues we must examine grows, and the eigenvalues' values change.

5. Conclusion:

The numerical outcomes demonstrate that the suggested Lines approach used to solve the diffusion problem yields accurate numerical solutions. No fictitious oscillations or instabilities were noticed throughout the tests that were conducted. Additionally, the results demonstrate that utilising higher Runge-Kutta methods does not necessarily "work harder" than using second-order Runge-Kutta to solve the system of ordinary differential equations.

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