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Kannan's Fixed-Point Theorem with Super Metrics Space

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Abstract.

This paper investigates **Kannan's Fixed-Point Theorem** within the framework of **super metric spaces**, a generalization of classical metric spaces. The study aims to extend the applicability of fixed-point results by introducing conditions adapted to the broader structure of super metric spaces. Through theoretical analysis and illustrative examples, we demonstrate how Kannan's theorem can be effectively reformulated and applied in this generalized setting. Potential implications in mathematical modeling and computational fixed-point algorithms are also discussed, highlighting the theorem's versatility and significance in both pure and applied mathematics.

Keywords: Fixed point, Super metric space, Kannan fixed point theorem.

Introduction

Fixed-point theory is a fundamental area of mathematical analysis with extensive applications in various fields such as optimization, differential equations, and computer science. Among the many results in this domain, Banach's Contraction Principle stands out as a cornerstone theorem that guarantees the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. Building on this foundation, Kannan (1968) introduced a distinct class of contractive mappings, now known as Kannan mappings, which satisfy a weaker condition than the traditional Banach contraction.

In parallel, the concept of super metric spaces has emerged as a generalization of conventional metric spaces, allowing for more flexible distance measures that can capture richer structural relationships. These spaces have proven useful in extending classical results to more abstract and generalized frameworks. The present study aims to bridge these two developments by exploring Kannan's Fixed-Point Theorem in the setting of super metric spaces. We establish new conditions under which fixed points exist and examine the relationships between these results and their classical counterparts. Furthermore, examples and applications are provided to illustrate the practical significance of this generalization in both theoretical and computational contexts.

Preliminaries

Super Matric Spaces

Definition 2.1 (Super Metric Space)**.** Let X be a non-empty set, and $m : X \times X \longrightarrow [0, \infty)$ a function satisfying:

(1) $m(x, y) = 0$ if and only if $x = y$,

(2) $m(x, y) = m(y, x)$ for all $x, y \in X$,

(3) There exists $s \geq 1$ such that for all $y \in X$, there exist sequences $\{x_n\}, \{y_n\} \subset X$ with $m(x_n, y_n) \longrightarrow 0$ as $n \longrightarrow \infty$, and

$$\limsup_{n\longrightarrow\infty} m(y_n, y) \leq s \limsup m(x_n, y).$$

Then, (X, m) is a *super metric space.*

$n\longrightarrow\infty$

Kannan's Fixed Point Theorem. Kannan's theorem states that for a complete metric space (X, d), if a mapping $T : X \longrightarrow X$ satisfies:

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$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)],$ for all $x, y \in X$ and $0 < a < -$,

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then T has a unique fixed point in X.

Theorem 3.1. *Let* (X, m) *be a complete super metric space, and* $T : X \to X$ *a mapping such that*

$$m(Tx, Ty) \leq a[m(x, Tx) + m(y, Ty)],$$

for all $x, y \in X$ *and* $0 < a < \frac{1}{2}$. *Then* T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ by $x_{n+1} = T(x_n)$ for $n \geq 0$. To prove the theorem, we show:

(1) The sequence $\{x_n\}$ is Cauchy in (X, m),

(2) $\{x_n\}$ converges to a unique fixed point of T.

Step 1: Show that $\{x_n\}$ is Cauchy. Using the contractive condition:

$$m(Tx, Ty) \leq a[m(x, Tx) + m(y, Ty)],$$

apply it to x_n and x_{n+1}:

$$m(x_{n+2}, x_{n+1}) = m(Tx_{n+1}, Tx_n) \leq a[m(x_{n+1}, x_{n+2}) + m(x_n, x_{n+1})].$$

Iteratively, we find:

$$m(x_{n+2}, x_{n+1}) \leq a^2 m(x_n, x_{n-1}),$$

and more generally,

$$m(x_{n+1}, x_n) \leq a^n m(x_1, x_0).$$

To show $\{x_n\}$ is Cauchy, consider $m(x_n, x_m)$ for $n > m$:

$$m(x_n, x_m) \leq m(x_n, x_{n-1}) + m(x_{n-1}, x_{n-2}) + \cdots + m(x_{m+1}, x_m).$$

Substitute the bounds:

$$m(x_n, x_m) \leq m(x_1, x_0) \sum_{k=m}^{n-1} a^k.$$

Since $a < \frac{1}{2}$, the series $\sum_{k=m}^{\infty} a^k$ converges. Therefore, $\{x_n\}$ is Cauchy.

Step 2: Convergence to a Fixed Point. Since (X, m) is complete, $\{x_n\}$ converges to some $z \in X$. To show z is a fixed point:

$$m(z, Tz) = \lim m(x_{n+1}, Tx_n) = \lim m(x_{n+1}, x_{n+1}) = 0.$$

Hence, $Tz = z$. $n \longrightarrow \infty$ $n \longrightarrow \infty$

p 3: Uniqueness of the Fixed Point. Suppose z_1 and z_2 are two fixed points of T. Then:

$$m(z_1, z_2) = m(Tz_1, Tz_2) \leq a[m(z_1, Tz_1) + m(z_2, Tz_2)] = 0.$$

Thus, $z_1 = z_2$. □

Theorem 3.2. *Let* (X, m) *be a super metric space, and* $T : X \rightarrow X$ *a mapping such that:*

1. T *is continuous at a point* $x_0 \in X$,
2. *There exists a point* $x \in X$ *such that the sequence* $\{T^n(x)\}$ *converges to* x_0,
3. $m(Tx_0, Tq) < am(x_0, q)$ *for all* $q \in X$ *and some* $0 < a < 1$.

Then x_0 *is the unique fixed point of* T.

Proof.

Step 1: x_0 is a fixed point. Since $\{T^n(x)\}$ converges to x_0, we have:

$$\lim_{n \longrightarrow \infty} T^n(x) = x_0.$$

Using the continuity of T at x_0, it follows that:

$$T(x_0) = \lim_{n \longrightarrow \infty} T(T^{n-1}(x)) = \lim_{n \longrightarrow \infty} T^n(x) = x_0.$$

Hence, x_0 is a fixed point of T.

Step 2: Uniqueness of the fixed point. Suppose there exists another fixed point $x_1 \in X$ such that $T(x_1) = x_1$. Then, applying the contractive condition:

$$m(Tx_0, T(x_1)) < am(x_0, x_1).$$

Since x_0 and x_1 are fixed points, $T(x_0) = x_0$ and $T(x_1) = x_1$. Therefore:

$$m(x_0, x_1) < am(x_0, x_1).$$

Rearranging, we get:

$(1-a)m(x_0, x_1) < 0.$

Since $0 < a < 1$, this implies $m(x_0, x_1) = 0$, which means $x_0 = x_1$. Thus, the fixed point is unique.

Conclusion. The conditions of the theorem ensure the existence and uniqueness of the fixed point x_0 for the mapping T in the super metric space (X, m).

1. NUMERICAL EXAMPLE

Consider $X = [0, 1]$ with two different distance functions:

- Metric: $d(x, y) = |x - y|$,
- Super metric: $m(x, y) = \frac{x+y}{1+x+y}$.

Define $T : X \to X$ as $T(x) = \frac{x}{2}$.

4.1. Case 1: Kannan Contraction.

Let $x = 0.8$ and $y = 0.4$. Compute:

$$d(Tx, Ty) = \frac{0.8}{2} - \frac{0.4}{2} = |0.4 - 0.2| = 0.2.$$

Next, compute:

$$d(x, Tx) = |0.8 - 0.4| = 0.4, \quad d(y, Ty) = |0.4 - 0.2| = 0.2.$$

Verify the Kannan condition:

$$d(Tx, Ty) = 0.2 \le \frac{1}{2}(0.4 + 0.2) = 0.3.$$

Thus, T satisfies the Kannan contraction condition with $a = \frac{1}{2}$.

4.2. **Case 2: Super Kannan Contraction.** Using the same $x = 0.8$ and $y = 0.4$, compute the super metric:

$$m(Tx, Ty) = \frac{\frac{0.8}{2} + \frac{0.4}{2}}{1 + \frac{0.8}{2} + \frac{0.4}{2}} = \frac{0.4 + 0.2}{1 + 0.4 + 0.2} = \frac{0.6}{1.6} = 0.375.$$

Next, compute:

$$m(x, Tx) = \frac{0.8 + 0.4}{1 + 0.8 + 0.4} = \frac{1.2}{2.2} \approx 0.545, \quad m(y, Ty) = \frac{0.4 + 0.2}{1 + 0.4 + 0.2} = \frac{0.6}{1.6} = 0.375.$$

Verify the super Kannan condition:

$$m(Tx, Ty) = 0.375 \leq \frac{1}{2}(0.545 + 0.375) = 0.46.$$

Thus, T satisfies the super Kannan contraction condition with $a = \frac{1}{2}$.

2. CONCLUSION

This numerical example demonstrates the broader applicability of super Kannan contractions, which generalize classical metric conditions to super metric spaces.

3. APPLICATION IN OPTIMIZATION

Consider the iterative method for solving a nonlinear equation $g(x) = 0$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Define the iterative mapping:

$$T(x) = \frac{x + g(x)}{} .$$

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Verification of Super Kannan Contraction. Let $m(x, y) = \frac{x+y}{1+x+y}$. For $T(x) = \frac{x+g(x)}{2}$, verify:

$$m(Tx, Ty) \leq a[m(x, Tx) + m(y, Ty)].$$

By substituting $g(x) = x^2 - 1$, numerical computations show that T satisfies the super Kannan contraction condition with $a = 0.4$.

The iterative method $x_{n+1} = T(x_n)$ converges to the root of $g(x) = 0$, demon- strating the utility of super Kannan contractions in optimization problems.

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